Geo-Metric-Affine-Projective Computing

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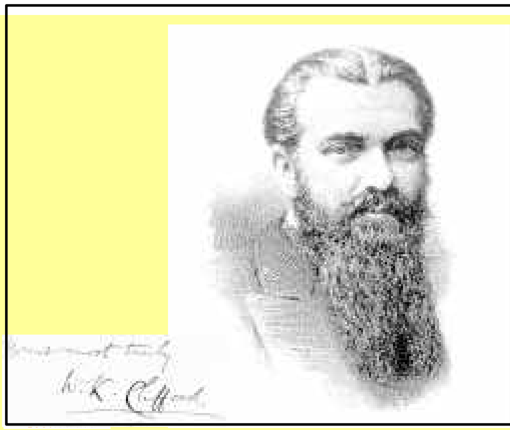
Projective Drawing Board (PDB)

PDB is an interactive program for doing plane projective geometry that will be used to illustrate this lecture.

PDB has been developed by
Harald Winroth and ***Ambjörn Naeve***
as a part of Harald's doctoral thesis work at the Computational Vision and Active Perception (CVAP) laboratory at KTH .

PDB is avaliable as freeware under Linux.
www.nada.kth.se/~amb/pdb-dist/linux/pdb2.5.tar.gz



Geometric algebra in *n*-dim Euclidean space

Underlying vector space $\mathbf{V}^n$ with ON-basis $e_1,\ldots,e_n$.

Geometric algebra: $\mathbf{G} = \mathbf{G}_n \equiv \mathbf{G}(\mathbf{V}^n)$ has 2^n dimensions.

A *multivector* is a sum of *k-vectors*: $M = \sum_{k=0}^{n} \langle M \rangle_k$

A *k-vector* is a sum of *k-blades*: $\langle M \rangle_k = A_k + B_k + \ldots$

A *k-blade* = blade of *grade k* : $B_k = b_1 \wedge b_2 \wedge \ldots \wedge b_k$

Note: $B_k \neq 0 \Leftrightarrow b_1,\ldots,b_k$ are linearly independent.

Hence: the grade of a blade is the dimension of the subspace that it spans.

Blades correspond to geometric objects

blade of grade	equivalence class of directed	equal orientation and
1	line segments	length
2	surface regions	area
3	3-dim regions	volume
⋮	⋮	⋮
k	*k*-dim regions	*k*-volume

Pseudoscalars and duality

Def: A n-blade in $\mathbf{G}_n$ is called a ***pseudoscalar***.

A pseudoscalar: $P = p_1 \wedge p_2 \wedge \ldots \wedge p_n$

A *unit* pseudoscalar: $I = e_1 \wedge e_2 \wedge \ldots \wedge e_n$

The *bracket* of P: $[P] = PI^{-1}$

The *dual* of a multivector x: $\mathrm{Dual}(x) = xI^{-1}$

Notation: $\mathrm{Dual}(x) = x^*$

Note: If A is a k-blade, then A^* is a $(n-k)$-blade.

The subspace of a blade

Fact: To every non-zero m-blade $B = b_1 \wedge \ldots \wedge b_m$ there corresponds a m-dim subspace $\bar{B} \subset \mathbf{V}^n$ with

$$\bar{B} = \mathrm{Linspan}\{b_1, \ldots, b_m\}$$
$$= \mathrm{Linspan}\{b \in \mathbf{V}^n : b \wedge B = 0\}.$$

Fact: If $e_1, e_2, \ldots, e_m$ is an ON-basis for $\bar{B}$ and if $b_i = \sum_{k=1}^{m} b_{ik} e_k$ for $i = 1,\ldots,m$,

then
$$B = (\det b_{ik}) e_1 \wedge e_2 \wedge \ldots \wedge e_m$$
$$= (\det b_{ik}) e_1 e_2 \ldots e_m .$$

Dual subspaces <=> orthogonal complements

Fact: If A is a non-zero m-blade $\overline{A^*} = \bar{A}^{\perp}$.

Proof: We can WLOG choose an ON-basis for $\mathbf{V}^n$ such that

$$A = l e_1 e_2 \ldots e_m \quad \text{and} \quad I = e_1 e_2 \ldots e_n .$$

We then have

$$A^* = AI^{-1} = \pm l e_{m+1} \ldots e_n$$

which implies that

$$\overline{A^*} = \bar{A}^{\perp}.$$

The join and the meet of two blades

Def: Given blades A and B, if there exists a blade C such that $A = BC = B \wedge C$ we say that A is a ***dividend*** of B and B is a ***divisor*** of A.

Def: The ***join*** of blades A and B is a *common dividend of lowest grade.*

Def: The ***meet*** of blades A and B is a *common divisor of greatest grade.*

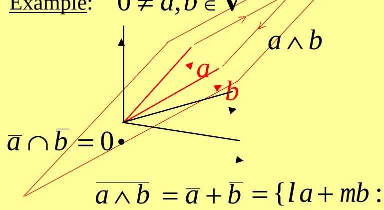
The join and meet provide a representation in geometric algebra of the ***lattice algebra*** of ***subspaces*** of $\mathbf{V}^n$.

Join of two blades <=> sum of their subspaces

Def: For two blades A and B with $A \wedge B \neq 0$ we can define: $\mathrm{Join}(A,B) = A \wedge B$.

In this case: $\overline{A \wedge B} = \bar{A} + \bar{B}$ and $\bar{A} \cap \bar{B} = 0$.

Example: $0 \neq a, b \in \mathbf{V}^3$



$$\overline{a \wedge b} = \bar{a} + \bar{b} = \{la + mb : l, m \in \mathrm{R}\}$$

Meet of blades <=> intersection of subspaces

Def: If blades $A, B \neq 0$ and $\bar{A} + \bar{B} = \mathbf{V}^n$

then: $\mathrm{Meet}(A,B) \equiv A \vee B = (A^* \wedge B^*)I$.

In this case: $\overline{A \vee B} = \bar{A} \cap \bar{B}$.

Note: The meet product is related to the outer product by duality:

$$(A \vee B)I^{-1} = (A^* \wedge B^*)II^{-1} = A^* \wedge B^*$$

$$\mathrm{Dual}(A \vee B) = \mathrm{Dual}(A) \wedge \mathrm{Dual}(B)$$

Dual outer product

Dualisation:

$$G \xrightarrow{*} G$$
$$x \mapsto x^* = xI^{-1}$$

Dual outer product:

$$\begin{array}{ccc} G\times G & \xrightarrow{\vee} & G \\ {}^*\downarrow\ \ {}^*\downarrow & & {}^*\downarrow \\ G\times G & \xrightarrow{\wedge} & G \end{array} \qquad \begin{array}{l} x\vee y = ((xI^{-1})\wedge(yI^{-1}))I \\ \\ x^*\wedge y^* = (x\vee y)^* \end{array}$$

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Example: $\mathbf{V}^3$, $I = e_1\wedge e_2\wedge e_3 = e_1e_2e_3$

$A = e_1\wedge e_2 = e_1e_2$, $B = e_2\wedge e_3 = e_2e_3$

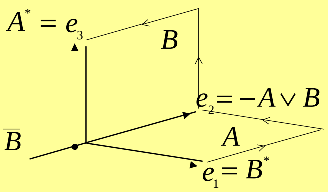
$A^* = (e_1\wedge e_2)I^{-1} = (e_1e_2)(e_3e_2e_1) = (-1)^2 e_3e_1e_2e_2e_1 = e_3$

$B^* = (e_2\wedge e_3)I^{-1} = (e_2e_3)(e_3e_2e_1) = e_1$

$A\vee B = (A^*\wedge B^*)I = (e_3\wedge e_1)(e_1e_2e_3) = -e_2$

Hence:

$\overline{A\vee B} = \overline{A}\cap\overline{B}$



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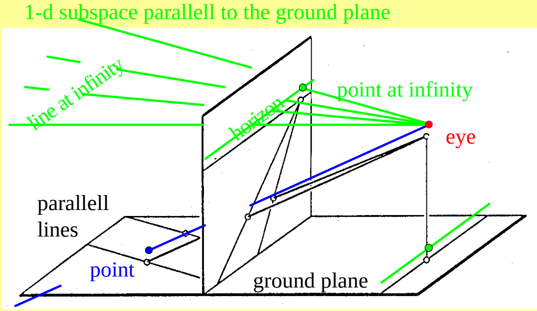
Projective geometry - historical perspective



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n-dimensional projective space $\mathbf{P}^n$

$\mathbf{P}^n = \mathbf{P}(\mathbf{V}^{n+1})$ = the set of non-zero subspaces of $\mathbf{V}^{n+1}$.

A *point* p is a 1-dim subspace (spanned by a 1-blade a).

$$p = \bar{a} = \{la : l \neq 0\} \cong a \cong la\ ,\ a\neq 0.$$

A *line* l is a 2-dim subspace (spanned by a 2-blade B_2).

$$l = \overline{B}_2 = \{lB_2 : l\neq 0\} \cong B_2.$$

Let $\mathbf{B}$ denote the set of non-zero blades of the geometric algebra $\mathbf{G}(\mathbf{V}^{n+1})$. Hence we have the mapping

$$\mathbf{B} \ni B \longmapsto \overline{B} \in \mathbf{P}^n.$$

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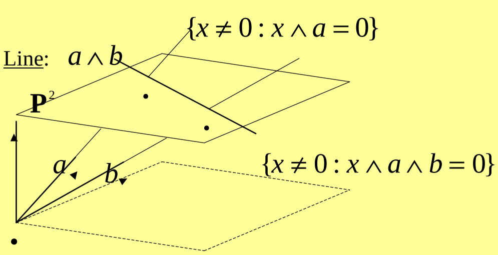
The projective plane $\mathbf{P}^2$

$0 \neq a, b, x \in \mathbf{V}^3$

Line: $a\wedge b$



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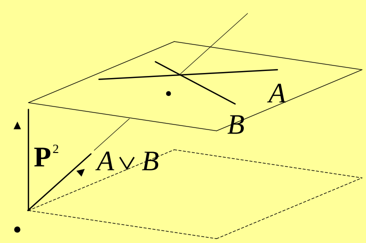
The intersection of two lines in $\mathbf{P}^2$

$A, B \in$ {non-zero 2-blades in $\mathbf{G}_3$}.



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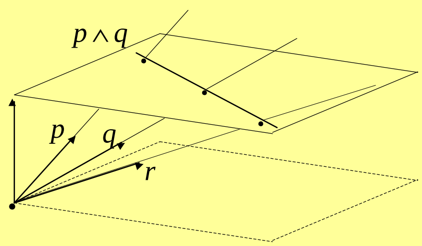
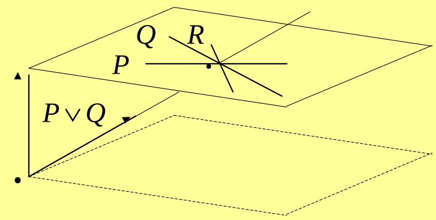
Collinear points



The points p, q, r are *collinear*

if and only if $p \wedge q \wedge r = 0$.

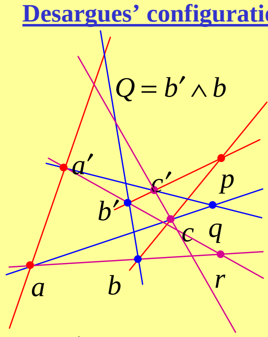
Concurrent lines



The lines P, Q, R are *concurrent*

if and only if $(P \vee Q) \wedge R = 0$.

Desargues' configuration



$A = b \wedge c$

$A' = b' \wedge c'$

$p = A \vee A'$

$B = c \wedge a$

$B' = c' \wedge a'$

$q = B \vee B'$

$C = a \wedge b$

$C' = a' \wedge b'$

$r = C \vee C'$

$P = a' \wedge a$ $\quad R = c' \wedge c$

Desargues' configuration (cont.)

$P = a' \wedge a$	$p = A \vee A' = (b \wedge c) \vee (b' \wedge c')$
$Q = b' \wedge b$	$q = B \vee B' = (c \wedge a) \vee (c' \wedge a')$
$R = c' \wedge c$	$r = C \vee C' = (a \wedge b) \vee (a' \wedge b')$

$J = a \wedge b \wedge c = [abc]I$

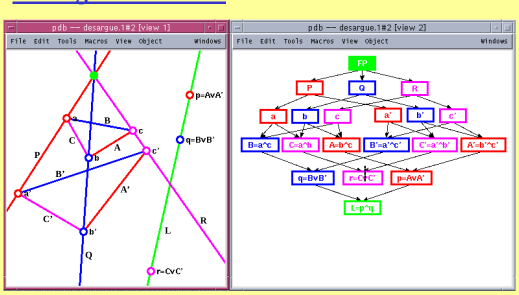
$J' = a' \wedge b' \wedge c' = [a'b'c']I$

Leads to: $\underbrace{p \wedge q \wedge r}_{} \equiv JJ'\underbrace{(P \vee Q) \wedge R}_{}$

= 0 if and only if p,q,r are collinear

= 0 if and only if P,Q,R are concurrent

Desargues' theorem



p, q, r are collinear iff P, Q, R are concurrent.

Pascal's theorem

Let a, b, c, a', b' be five given points in $\mathbf{P}^2$.

Consider the second degree polynomial given by

$$p(x) = ((a \wedge b') \vee (a' \wedge b)) \wedge ((b \wedge x) \vee (b' \wedge c)) \wedge ((c \wedge a') \vee (x \wedge a)).$$

It is obvious that $p(a) = p(b) = 0$

and easy to verify that $p(c) = p(a') = p(b') = 0$.

Hence: $p(x) = 0$ must be the equation of the conic on the 5 given points.

Verifying that $p(a') = 0$

$p(a') =$

$$= (\overbrace{(a\wedge b')\vee(a'\wedge b)}^{\text{same line}}) \wedge (\overbrace{(b\wedge a')\vee(b'\wedge c)}^{\text{same line}})\wedge$$

$$Ÿ a' \wedge b$$

$$((\overset{\text{same point}}{c \wedge a')\vee(a' \wedge a)}) \; Ÿ (a'\wedge b)\wedge a' = 0$$

$$Ÿ a'$$

Pascal's theorem (cont.)

Hence, a sixth point c' lies on this conic if and only if

$$p(c') = ((a\wedge b')\vee(a'\wedge b))\wedge$$
$$((b\wedge c')\vee(b'\wedge c))\wedge$$
$$((c\wedge a')\vee(c'\wedge a)) = 0.$$

Geometric formulation:

> The three points of intersection of opposite sides of a hexagon inscribed in a conic are collinear.

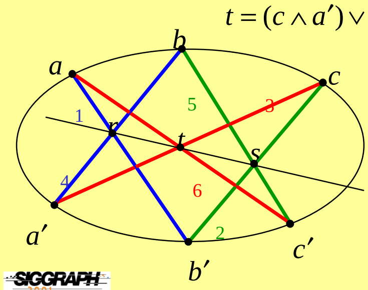
This is a property of the ***hexagrammum mysticum***, which Blaise Pascal discovered in 1640, at the age of 16.

Pascal's theorem (cont.)

$r = (a\wedge b')\vee(a'\wedge b)$ = 1 ı 4

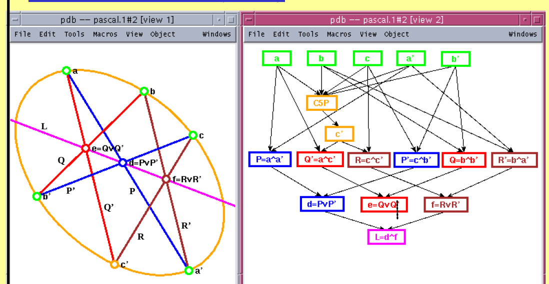
$s = (b\wedge c')\vee(b'\wedge c)$ = 2 ı 5

$t = (c\wedge a')\vee(c'\wedge a)$ = 3 ı 6



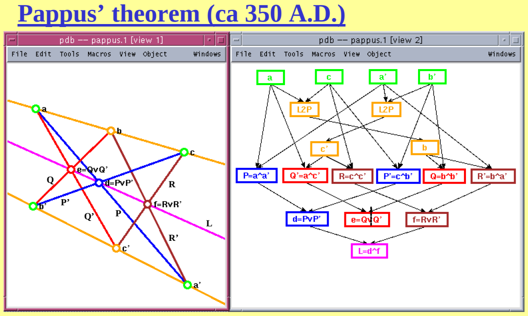
$r\wedge s\wedge t = 0$

Pascal's theorem (cont.)



The three points of intersection of opposite sides of a hexagon inscribed in a conic are collinear.

Pappus' theorem (ca 350 A.D.)



If the conic degenerates into two straight lines, ***Pappus' theorem*** emerges as a special case of Pascal's.

Outermorphisms

Def: A mapping $f : \mathbf{G} \to \mathbf{G}$ is called an *outermorphism* if

(i) f is linear.

(ii) $f(t) = t \quad \forall t \in \mathbf{R}$

(iii) $f(x\wedge y) = f(x)\wedge f(y) \quad \forall x,y \in \mathbf{G}$

(iv) $f(\mathbf{G}^k) \subset \mathbf{G}^k \quad \forall k \geq 0$

The induced outermorphism

Let $T : \mathbf{V} \rightarrow \mathbf{V}$ denote a linear mapping.

Fact: T induces an ***outermorphism*** $\underline{T} : \mathbf{G} \rightarrow \mathbf{G}$ given by

$$\underline{T}(a_1 \wedge \ldots \wedge a_k) = T(a_1) \wedge \ldots \wedge T(a_k)$$
$$\underline{T}(l) = l \ , \ l \in \mathbf{R}$$

and linear extension.

Interpretation: $\underline{T}$ maps the *blades* of $\mathbf{V}$ in accordance with how T maps the *vectors* of $\mathbf{V}$.

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Polarization with respect to a quadric in $\mathbf{P}^n$

Let $T : \mathbf{V}^{n+1} \rightarrow \mathbf{V}^{n+1}$ denote a *symmetric* linear map, which means that $T(x) \cdot y = x \cdot T(y) \ , \forall x, y \in \mathbf{V}^{n+1}$.

The corresponding quadric (hyper)surface Q in $\mathbf{P}^n$ is given by $Q = \{x \in \mathbf{V}^{n+1} : x \cdot T(x) = 0 \ , \ x \neq 0\}$.

Def: The *polar* of the k-blade A with respect to Q is the (n+1-k)-blade defined by

$$\text{Pol}_Q(A) \equiv \underline{T}(A)^* \equiv \underline{T}(A) I^{-1}.$$

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Polarization (cont.)

Note: $T = id \Rightarrow Q = \{x \in \mathbf{V}^{n+1} : x \cdot x = 0 \ , \ x \neq 0\}$.
In this case $\text{Pol}_Q(A) \equiv AI^{-1} \equiv A^*$
and polarization becomes identical to dualization.

Fact: For a blade A we have

(i) $\text{Pol}_Q(\text{Pol}_Q(A)) \simeq A$

(ii) If A is tangent to Q then $\text{Pol}_Q(A)$ is tangent to Q.

Especially: If x is a point on Q, then $\text{Pol}_Q(x)$ is the hyperplane which is tangent to Q at the point x.

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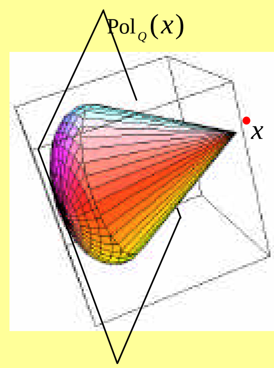

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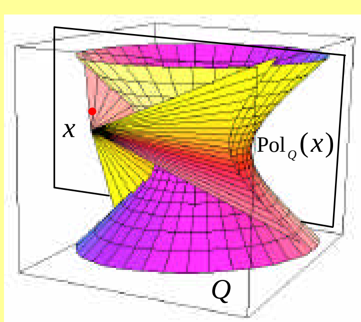

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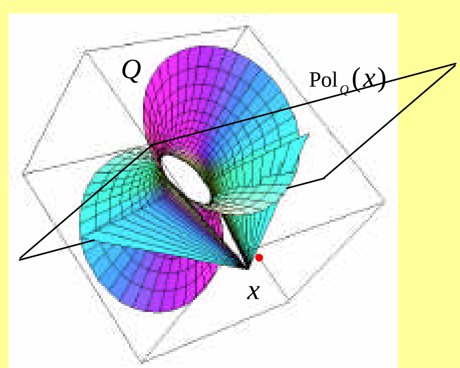

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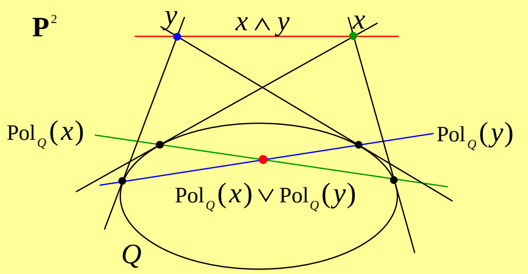
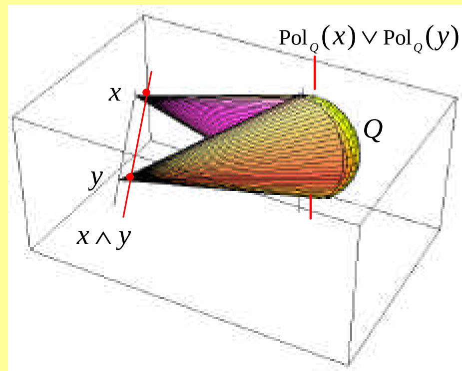
Polarity with respect to a conic



The polar of the ***join*** of x and y = $\mathrm{Pol}_Q(x \wedge y)$
is the ***meet*** of the polars of x and y = $\mathrm{Pol}_Q(x) \vee \mathrm{Pol}_Q(y)$



Polar reciprocity

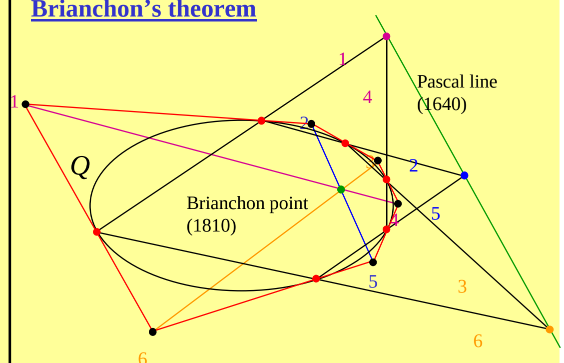
Let $x, y \in \mathbf{V}^{n+1}$ represent two points in $\mathbf{P}^n$.
Then we have from the symmetry of T :

$$
\begin{aligned}
y \wedge T(x)^* &= y \wedge (T(x)I^{-1}) \\
&= (y \cdot T(x))I^{-1} = (x \cdot T(y))I^{-1} \\
&= x \wedge (T(y)I^{-1}) = x \wedge T(y)^*.
\end{aligned}
$$

Hence: $y \wedge \mathrm{Pol}_Q(x) = 0 \iff x \wedge \mathrm{Pol}_Q(y) = 0$

i.e. the point y lies on the polar of the point x
if and only if x lies on the polar of y.

Brianchon's theorem



The dual map

Let $f : \mathbf{G} \to \mathbf{G}$ be linear,
and assume that $I^2 \neq 0$.

Def: The dual map $\tilde{f} : \mathbf{G} \to \mathbf{G}$
is the linear map given by

$$\tilde{f}(x) = f(xI)I^{-1}$$

$$
\begin{tikzcd}
\mathbf{G} \arrow[r, "f"] \arrow[d, "*"'] & \mathbf{G} \arrow[d, "*"] \\
\mathbf{G} \arrow[r, "\tilde{f}"] \arrow[d, "*"'] \arrow[dr, phantom, "\tilde{f}(\tilde{x})=f(\tilde{x})"] & \mathbf{G} \arrow[d, "*"] \\
\mathbf{G} \arrow[r, "\tilde{\tilde{f}}=f"] & \mathbf{G}
\end{tikzcd}
$$

Note: $f(x) = \tilde{f}(xI^{-1})I = \tilde{f}(xI^{-1})I^2I^{-2}I$
$= \tilde{f}(xI^{-1}I^2)I^{-1} = \tilde{f}(xI)I^{-1} = \tilde{\tilde{f}}(x)$

Polarizing a quadric with respect to another

Let $S : \mathbf{G} \to \mathbf{G}$ and $T : \mathbf{G} \to \mathbf{G}$
be symmetric outermorphisms, and let

$$P = \{x \in \mathbf{G} : x * S(x) = 0 \;,\; x \neq 0\},$$
$$Q = \{x \in \mathbf{G} : x * T(x) = 0 \;,\; x \neq 0\}$$

be the corresponding two quadrics.

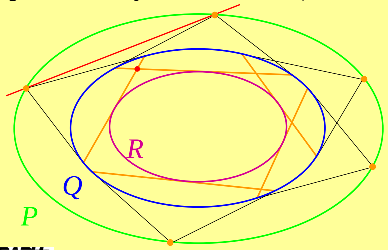
Fact: Polarizing the multivectors of the quadric P
with respect to the quadric Q
gives a quadric R
with equation $x * (T \circ \tilde{S} \circ T(x)) = 0$
and we have $x \in P \iff \mathrm{Pol}_Q(x) \in R$.

The reciprocal quadric

Polarizing the quadric $P: \; x * S(x) = 0$

with respect to the quadric $Q: \; x * T(x) = 0$

generates the quadric $R: \; x * (T \circ \tilde{S} \circ T(x)) = 0$



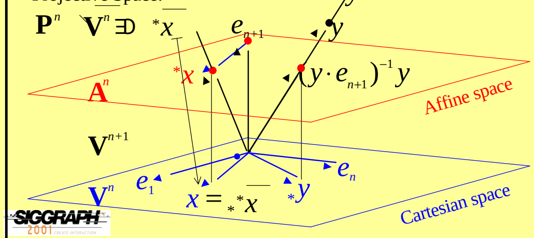
Cartesian-Affine-Projective relationships

$$\mathbf{V}^n \ni x \longmapsto {}^*x = x + e_{n+1} \in \mathbf{A}^n$$

$$\mathbf{V}^{n+1} \supset \mathbf{V}^n \ni y \longmapsto {}_*y = (y \cdot e_{n+1})^{-1} y - e_{n+1} \in \mathbf{V}^n$$

The affine part of Projective Space: $\mathbf{P}^n \setminus \overline{\mathbf{V}^n} \ni {}^*\overline{x}$

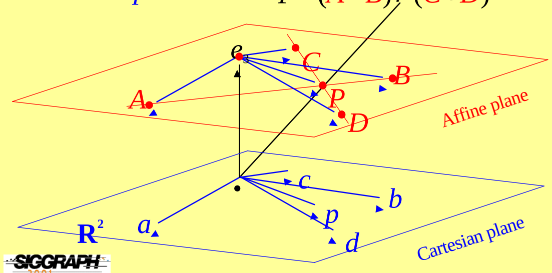
well defined $\quad \overline{y} \in \mathbf{P}^n \setminus \overline{\mathbf{V}^n}$



The intersection of two lines in the plane

$$a,b,c,d \xrightarrow{\;{}^*(\,)\;} A,B,C,D$$

$$p \xleftarrow{\;{}_*(\,)\;} \overline{P} = (A \wedge B) \vee (C \wedge D)$$



The intersection of two lines (cont.)

$$\overline{P} = (A \wedge B) \vee (C \wedge D)$$

$$= [A \wedge B \wedge C\,]D - [A \wedge B \wedge D\,]C = aD - bC$$

$$[A \wedge B \wedge C\,] = (A \wedge B \wedge C)I^{-1}$$

$$= \big((a+e_3) \wedge (b+e_3) \wedge (c+e_3)\big)e_3e_2e_1$$

$$0 = \quad = \big((a-c) \wedge (b-c) \wedge (c+e_3)\big)e_3e_2e_1$$

$$= \big(\boxed{(a-c) \wedge (b-c) \wedge c}\big)e_3e_2e_1 +$$

$$\big((a-c) \wedge (b-c) \wedge e_3\big)e_3e_2e_1$$

does not contain e_3

$$= \big(\boxed{(a-c) \wedge (b-c)}e_3\big)e_3e_2e_1$$

$$= \boxed{(a-c) \wedge (b-c)e_2e_1} \equiv a$$

The intersection of two lines (cont.)

In the same way we get

does not contain e_3

$$b = [A \wedge B \wedge D\,] = (A \wedge B \wedge D)I^{-1}$$

$$= \boxed{(a-d) \wedge (b-d)e_2e_1}$$

Hence we can write p as:

$$p = {}_*\overline{P} = {}_*(aD - bC)$$

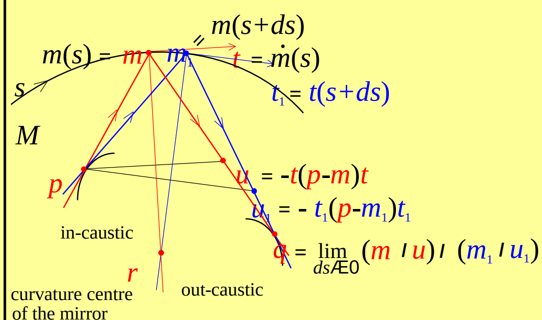
$$= ((aD - bC) \cdot e_3)^{-1}(aD - bC) - e_3$$

$$= (a-b)^{-1}(ad + ae_3 - bc - be_3) - e_3$$

$$= (a-b)^{-1}(ad - bc)$$

does not contain e_3

Reflection in a plane-curve mirror



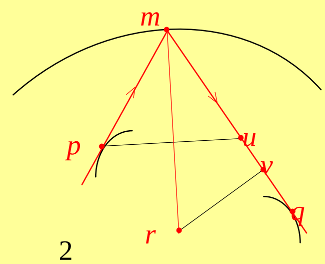
Reflection in a plane-curve mirror (cont.)

Making use of the intersection formula deduced earlier and introducing $n = \frac{\dot{t}}{|\dot{t}|}$ for the unit mirror normal we get

$$q - m = \frac{((p-m)\cdot t)t - ((p-m)\cdot n)n}{1 - 2|\ddot{m}|\dfrac{(p-m)^2}{(p-m)\cdot n}}$$

This is an expression of Tschirnhausen's reflection law.

Reflection in a plane-curve mirror (cont.)



Tschirnhausens reflection formula

$$\frac{1}{|u-m|} \pm \frac{1}{|q-m|} = \frac{2}{|v-m|}$$

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Winroth, H., *Dynamic Projective Geometry*, TRITA-NA-99/01, ISSN 0348-2953, ISRN KTH/NA/R--99/01--SE, Dissertation, The Computational Vision and Active Perception Laboratory, Dept. of Numerical Analysis and Computing Science, KTH, Stockholm, March 1999.